Stability of topologically invariant order parameters at finite temperature

Isaac H. Kim[1]

[1]Institute of Quantum Information, California Institute of Technology, Pasadena CA 91125, USA

(Dated: September 19, 2011)

Topological entanglement entropy is a topological invariant which can detect topological order of quantum many-body ground state. We assume an existence of such order parameter at finite temperature which is invariant under smooth deformation of the subsystems, and study its stability under hamiltonian perturbation. We apply this assumption to a Gibbs state of hamiltonian which satisfies so called 'strong commuting' condition, which we shall define in the paper. Interesting models in this category include local hamiltonian models based on quantum error correcting code. We prove a stability of such topologically invariant order parameter against arbitrary perturbation which can be expressed as a sum of geometrically local bounded-norm terms. The first order correction against such perturbation vanishes in the thermodynamic limit.

In a quantum many-body system at zero temperature, system can exhibit topological order depending on the parameters of the hamiltonian. Prototypical examples include Kitaev's quantum double model and Levin-Wen string-net model.[1,2] Unfortunately the interaction terms of these models are usually more than two body. This might be problematic, since it is hard to find materials which can be described exactly by such models. One may hope, however, that these hamiltonian arises as an low-energy effective hamiltonian of a simpler model. If the perturbative terms in the effective hamiltonian is small enough, one can argue that the topological order may still be protected.[3–5] For gapped frustration-free hamiltonian, certain topological order condition on the ground state ensures a gap protection against geometrically local bounded norm perturbation.

It is plausible to conclude gap protection implies the stability of topological order, since absence of quantum phase transition is likely to be an evidence that they belong to the same phase. This logic cannot be applied to finite temperature, since the density operator has nontrivial support on excited states as well. This problem is not so interesting in 2D, since there is no finite temperature topological order in 2D.[6–8] However, nontrivial topological order might survive at finite temperature for higher dimensional system. For instance, 3D generalization of Levin-Wen type topological entanglement entropy is nonzero below the critical temperature of 3D toric code.[9] Other models with similar structure will likely show a similar behavior.[10–15] This type of topological order is of classical origin. Even though the ground state of the system encodes nontrivial quantum information, only classical information survives at finite temperature. 4D toric code on the other hand can successfully preserve quantum information even at finite temperature.[16]

One of the main objectives of this paper is to establish an analogous stability result at finite temperature, potentially to the systems discussed above. We have approached this problem by observing the common properties of these systems. When the system is topologically ordered, there is a nontrivial topological entanglement entropy. Important property of topological entanglement entropy is that it is invariant under small deformation of the subsystems as long as they do not change the 'shape' of the configuration. When the system is in a topologically disordered state, topological entanglement entropy is likely to be 0, as evidenced in the calculation of 3D toric code.[9] The topological entanglement entropy of this system is still 'invariant' in a sense that it stays 0 with a small deformation of the subsystem. In an actual model, such invariance is more likely to be an approximation of a small number that vanishes in the thermodynamic limit.

The paper can be roughly divided into two parts. First part is about studying the implications of such topological invariance. We shall present a general argument that such systems must not have any long range order. Furthermore, we shall show that such invariance implies an *asymptotic conditional independence* for certain configurations. A tripartite state ρ_{ABC} is conditionally independent if a conditional mutual information $I(A:B|C)$ is 0. Asymptotic conditional independence means $I(A:B|C)=\epsilon$ for some ϵ that vanishes in the thermodynamic limit. It is important to note that asymptotic conditional independence does not hold for arbitrary configurations. We shall study the configurations arising from the invariance of topological entanglement entropy under smooth deformation.

The rest of the paper is about showing the implication of asymptotic conditional independence. We shall define a class of models which satisfy the condition called *strong commuting condition.* For such models, one can bound a first-order perturbation of Levin-Wen type topological entanglement entropy under a sum of geometrically local finite-norm perturbations. The implication of this result is twofold. If there exists a topological order in a sense of having nonzero topological entanglement entropy, it is stable against hamiltonian perturbation. Hence the topological state of matter is robust. On the other hand, it is impossible to create topological order from a disordered system by adding a small perturbation.

The proof depends on two statements about quantum many-body systems. The first statement is a certain variant of Lieb-Robinson bound. This technique was used by Hastings in proving certain locality properties of fi-

nite temperature quantum systems.[17,18] Second statement concerns a spectrum of an operator whose trace reduces to conditional mutual information. The second statement is the only part where the strong commuting condition is needed. Logic of the rest of the paper remains intact as long as the hamiltonian can be expressed as a sum of geometrically local bounded-norm terms. We shall discuss how the second statement may potentially be generalized to a wider class of models.

Once these technical results are established, the conceptual idea behind the proof is quite simple. Effect of the perturbation by local terms can be written as a sum of various correlation functions. These terms can be bounded by conditional mutual information, which was assumed to be small. We set the stage by motivating and precisely formulating the asymptotic conditional independence condition for topologically ordered system in Section I. We sketch proof and present the mathematical ideas in Section II. The main stability result is stated in III. Section IV will discuss how this result can be applied in the contexts of known models.

I. ENTANGLEMENT ENTROPY OF TOPOLOGICALLY ORDERED SYSTEM

Ground state of topologically ordered system satisfies an area law.

$$S_A = c|\partial A| - \gamma + o(1), \quad (1)$$

where γ is a quantum dimension of the system. It is usually implicitly assumed that A is a simply connected subsystem of the entire lattice Λ. If A changes into a topologically distinct object, γ should change as well. For instance, if A is a union of two simply connected subsystems which are widely separated, γ should be replaced by 2γ. This tells us that γ must be not only a function of the state, but also a function of the topology of A, hence we write it instead as γ_A from now on. Unfortunately topology in the spirit of classifying shapes of smooth objects are not a well defined concept for discrete systems. For instance, consider a torus represented as $n \times n$ grid, both ends identified. Open string can be mapped into a closed string and vice versa via 'smooth deformation,' which corresponds to adding or subtracting a single cell. One remedy for such problem is to embed the entire system onto a manifold and designate a finite neighborhood near each particles, so that the finite neighborhoods partition the entire manifold. Under this embedding, one can easily see that open string and closed string have clear topological meaning: 'smooth deformation' once lifted to the entire manifold cannot map open string into closed string and vice versa. This is the setting in which we shall explain our result. When we say subsystem A, it shall mean a subset of the manifold which is a union of finite neighborhoods near each particle contained in A. Similarly, when we say an area and volume of A, it shall mean the area and volume of the embedding on the manifold.

If γ_A only depends on the topology of A, it must be equal to $\gamma_{A'}$ for two topologically equivalent subsystems $A \sim A'$. The easiest nontrivial implication can be seen in FIG.1. For each configurations, γ_A is either γ or 2γ, depending on if it is a simply connected region or union of two such widely separated subsystems. For all of these configurations, $I(A : B|C) = o(1)$.

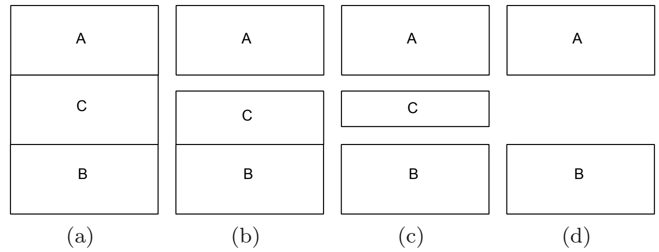


FIG. 1: Three possible configurations of A, B and C that produces $I(A : B|C) = o(1)$. When the subsystems are contingent to each other, they share a boundary. Otherwise, they are assumed to be far away from each other.The last figure denotes $C = \emptyset$.

In fact, knowing the exact value of γ_A is unimportant for showing such cancelations as long as we can find a pair of shapes which are topologically equivalent to one another. For instance, consider FIG.2. The constant subcorrection term of S_{AB} and S_B are both γ, so they cancel out each other trivially. ABC and BC are both a closed surface with a puncture. Since their shapes are topologically equivalent, the constant subcorrection term cancel out. One can easily check that the area term cancels out each other as well.

Equation 1 is a strong evidence that degrees of freedom is localized near the boundary of A. In general, what is more likely to happen is that the majority of the degrees of freedom are located near the boundary and there are small tails that decay as we move away. Due to this decay, tails are negligible when A and C are far away from

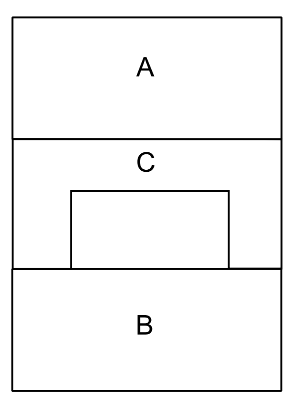


FIG. 2: Another configuration which yields $I(A : B|C) = o(1)$.

each other, but it may not be when they are close together. From these observations, we arrive at the precise definition of asymptotic conditional independence.

Assumption 1. *If i)$AB \sim B$ and $BC \sim ABC$ or ii)$AB \sim ABC$ and $BC \sim B$, $I(A : B|C) \leq f(l_{AC})$ for some decaying function f, where l_{AB} is the distance between A and B.*

Volume term emerges at finite temperature, but the cancelation property remains the same: $Vol(AB) + Vol(BC) - Vol(ABC) - V(C) = 0$ when A, B and C only coincide on the boundary. Assumption 1 implicitly assumes a finite correlation length. Hence we do not expect it to hold for critical systems.

If the configurations for extracting the topological entanglement entropy is known, there is an alternative way to see the asymptotic conditional independence. Consider a Levin-Wen type configuration or its 3D generalization. Topological entanglement entropy can be written as $I(A : B|C) = 2\gamma$. If this quantity is a topological invariant, it must remain the same under a deformation that preserves the shape of the subsystems. Simple algebraic manipulation shows $I(AD : B|C) - I(A : B|C) = I(D : B|AC)$. We have a small number on the left hand side, since the assumption was the invariance of topological entropy under small deformations. This may not be true when D is close to B. Hence right hand side must be a function that primarily depends on the distance between D and B which converges to 0. Notice that the configurations arising from $I(D : B|AC)$ in FIG.3 are topologically equivalent to configurations arising from $I(A : B|C)$ in FIG.2. Since 3D generalization of Levin-Wen configuration can be also written as $I(A : B|C)$ for some configurations, same logic can be applied as well. Notice that we did not assume anything about the specifics of the UV-divergent terms of entanglement entropy. If the topological entanglement entropy is invariant under smooth deformation of the subsystems, conditional mutual information must be a small number in the thermodynamic limit.

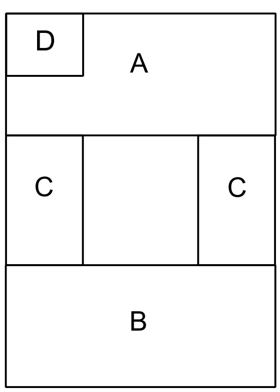


FIG. 3: Levin-Wen construction for topological entanglement entropy and its deformation.

These two different perspectives have their own caveat. If we assume certain cancelation property of UV-divergent terms, the asymptotic conditional independence condition is obtained. On the other hand, assuming the existence of certain topologically invariant order parameter automatically implies the same result irregardless of the specifics about the entanglement entropy. It is not entirely clear, however, that if there always exists such order parameter. Furthermore, the smallness of $I(A : B|C)$ for configuration in FIG.1(d) is essential for our proof. One cannot derive it solely based on the topological invariance of certain order parameter. This is still a plausible assumption nonetheless, since the system likely possesses a finite correlation length in noncritical systems.

Either way, these evidences strongly support the validity of asymptotic conditional independence for topologically ordered system. Remainder of this paper is about studying the consequence of this statement, but we need to be specific. We are trying to bound a first-order perturbation of topological entanglement entropy under a sum of geometrically local finite-norm terms, but what kind of topological entanglement entropy should we use? Levin-Wen type configuration and its generalization seem to work, but the proof is rather general. For this reason, we would like to start with an abstract definition, and observe how it can be applied to the configurations studied in the literature.

We shall name linear combination of entropies in general as *entropic order parameter* $\mathcal{S}(\{A_i\}) = \sum a_i S_{A_i}$. Spatial deformation of the order parameter can be written as $\mathcal{S}(\{A_i\}) - \mathcal{S}(\{A_i'\})$ in general for some deformed subsystem A_i's. We are interested in a following class of entropic order parameters.

Definition 1. *An entropic order parameter $\mathcal{S}(\{A_i\})$ is (l_1, l_2)-deformable if for all local site B, there exists deformation $\{A_i\} \to \{A_i'\}$ such that*

$$\mathcal{S}(\{A_i\}) - \mathcal{S}(\{A_i'\}) = \sum_j a_j I(j_1 : j_2|j_3), \qquad (2)$$

for $dist(j_1, j_2) \geq l_1$, $\min dist(A_i', B) \geq l_2$. $\{a_j\}$ is some sequence of numbers.

Important property of the order parameters satisfying this definition is that one can always deform away the order parameter from any local site with distance at least l_1. As we shall see, effect of perturbation can be written as a sum of thermal correlation function between two observables. For large enough distance, thermal correlation between the local site and the deformed order parameter becomes negligible due to the existence of finite correlation length. What remains to be seen is the thermal correlation between the difference between these two order parameters and the local site. There is no guarantee that the distance between those two observables to be large in general, but we shall show that it is possible to bound it nonetheless from conditional mutual information. The

perturbation bound vanishes in the limit $l_1, l_2 \to \infty$, so one must check if the order parameter is l_1, l_2-deformable for large enough value of l_1 and l_2.

A. 2D and 3D Levin-Wen construction

First example of entropic order parameter which allows a large value of l_1 and l_2 is Levin-Wen configuration in 2D. Assume A and B are $3R \times R$ rectangle, and C a union of two $R \times R$ squares. We can show that topological entanglement entropy $I(A:B|C)$ is $(\frac{R}{4}, \frac{R}{4})$-deformable. Let s denote a local site. If s is distance $\frac{R}{4}$ or larger away from ABC, the conditions are automatically satisfied. Let the distance from s to A, B, C be d_A, d_B, d_C. At least one of d_A or d_B must be larger than $\frac{R}{2}$. Without loss of generality, assume $d_A \leq d_B$. First consider deforming A into $A \setminus D$. Using $I(A:B|C) - I(A \setminus D : B|C) = I(D:B|AC \setminus D)$. For any point, it is possible to choose D such that the distance between s and $A \setminus D$ is larger than $\frac{R}{4}$. For the distance between C and the s, the effect of deformation can be written as $I(D:B|AC \setminus D) - I(D:B|C \setminus D)$. Again, it is possible to choose D such that the distance between s and $C \setminus D$ is larger than $\frac{R}{4}$. An example of such deformation is illustrated in FIG.4.

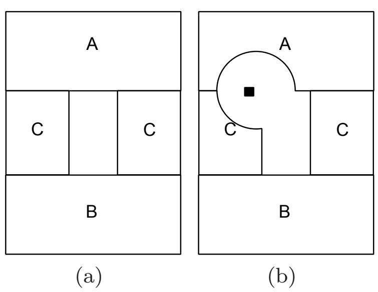


FIG. 4: Configurations before and after the deformation. Black colored region denotes the position of the local operator. Irregardless of the position of this operator, one can always deform the original operator so that it can be seaparted from the local operator.

Similar idea can be applied to 3D generalization of Levin-Wen construction. See FIG.5. As in the 2D case, one can deform away the region around a local site. We did not denote the position of the local operator due to the clarity of the picture, but it is located near the center of cubic deformation shown in FIG.5(e), 5(f), 5(g), 5(h). There is an alternative 3D generalization for Levin-Wen model. This can be easily reconstructed by stacking slices of FIG.4 together. Same logic can be applied here as well.

B. Alternative configurations

Recently Grover et al. introduced a number of ways to extract the constant subcorrection term of entanglement entropy in 3D.[19] These are generally a 3D generalization

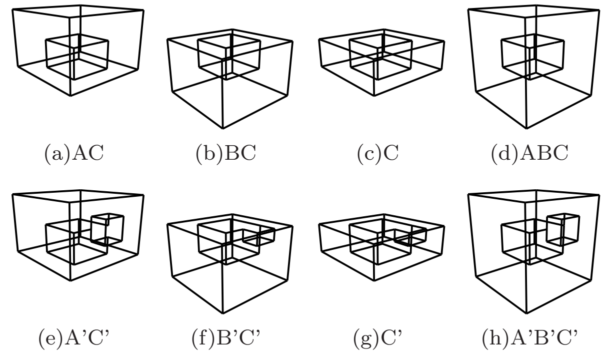


FIG. 5: One possible generalization of Levin-Wen construction in 3D. Topological entanglement entropy γ can be extracted by $I(A:C|B) = 2\gamma$. The first four configurations denote the original configuration. The next four configurations denote the configurations after the deformation.

of Kitaev-Preskill construction in some ways. This means the topological entanglement entropy is extracted by a formula $S_{topo} = S_A + S_B + S_C - S_{AB} - S_{BC} - S_{AC} + S_{ABC}$ for some configuration A, B, and C. One of their proposal, which is a three equipartition of torus allows a (l_1, l_2)-deformability for large values of l_1 and l_2.

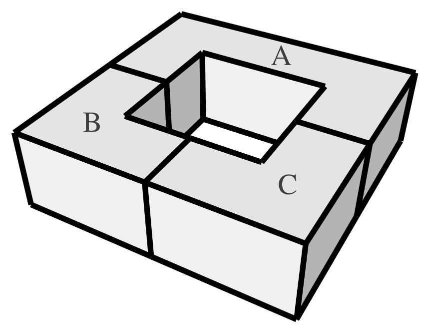
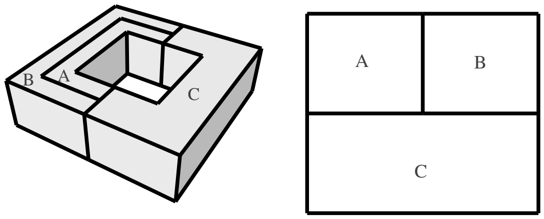


FIG. 6: Alternative 3D configuration that allows deformability for large values of l_1, l_2.

C. Some configurations are not deformable

In showing the deformability of the configuration, it was vital to assume that at most two subsystems involved in the computation of entropic order parameter must be contingent on each local sites. There are several configurations which are known in the literature to disobey this rule. Primary example is Kitaev-Preskill construction in 2D: there exists a triple point where three of the subsystems involved in the computation of entropic order parameter meet together. One of Grover et al.'s construction also share a similar property. These are depicted in

FIG.7. These are naturally excluded from the scope of the stability proof.



(a) (b)

FIG. 7: Configurations which are not deformable with large values of l_1, l_2.

II. TECHNICAL RESULTS

There are two main technical results that deserve to be discussed separately. First is a variant of Lieb-Robinson bound. Consider an infinitesimal perturbation V on the hamiltonian H, $H \to H + \epsilon V$. Such perturbation can be characterized by a directional derivative of H.

Definition 2.

$$\partial_H^V f(H) = \lim_{\epsilon \to 0} \frac{f(H+\epsilon V) - f(H)}{\epsilon}. \tag{3}$$

If the perturbation V is local, Hastings showed that

$$\partial_H^V e^{-\beta H} = \frac{1}{2}\beta(V' e^{-\beta H} + e^{-\beta H} V'), \tag{4}$$

for some quasi-local operator V'.[17,18] In the eigenbasis of H, V' can be expressed as the following.

$$V'_{ij} = V_{ij} \frac{\tanh x_{ij}}{x_{ij}}, \tag{5}$$

with $x_{ij} = \frac{\beta(E_j - E_i)}{2}$. E_is are the eigenvalues of H. Immediate application of this formula concerns an effect of local perturbation on the thermal expectation value of some observable.

$$\partial_H^V \langle \sigma \rangle = Re(\mathcal{C}(V', \sigma)), \tag{6}$$

where $\mathcal{C}(O_1, O_2) = \langle O_1 O_2 \rangle - \langle O_1 \rangle \langle O_2 \rangle$ is a connected correlation function. One of the general recurring themes throughout this paper is that V' and σ are observables which are far away from each other. In those cases, it is possible to bound the connected correlation function from mutual information. Locality of V' can be trivially shown for hamiltonian which consist of geometrically local commuting terms, but it is also important to note that its operator norm is bounded by $\|V'\| \leq \|V\|$. This is explained in Appendix A. In this context, locality of V' can be stated in a following way.

Lemma 1. *Let r be the maximal radius of interaction of the local terms in H. If the local terms commute with each other,*

$$[V', O] = 0 \tag{7}$$

for $dist(supp(O), supp(V)) > r$. $supp(O)$ is the minimal nontrivial support of O.

The second idea concerns a special property of conditional mutual information. Note that conditional mutual information $I(A:B|C)$ can be written in a following form.

$$I(A:B|C) = tr[\rho \hat{H}_{A:B|C}], \tag{8}$$

with $\hat{H}_{A:B|C} = \ln \rho_{ABC} + \ln \rho_B - \ln \rho_{AB} - \ln \rho_{BC}$. $\ln \rho_A$ actually means $\ln(\rho_A \otimes I_{A^c}) = \ln \rho_A \otimes I_{A^c}$. Consider an observable σ whose support lies on BC. For a classical system,

$$|Re(tr[\rho\sigma \hat{H}_{A:B|C}]| \leq I(A:B|C)\|\sigma\|. \tag{9}$$

One way to see this is to take a partial trace over subsystem A. σ is unaffected since it does not have nontrivial support on A. Using the fact that $\rho_{ABC} = \rho_{A|BC}\rho_{BC}$ and the positivity of relative entropy $D(\rho_{A|BC}\|\rho_{A|B})$, one can show that $tr_A[\rho_{ABC}\hat{H}_{A:B|C}] \geq 0$. Hence

$$\begin{aligned} |Re(tr[\rho\sigma\hat{H}_{A:B|C}])| &\leq |tr_A[\rho_{ABC}\hat{H}_{A:B|C}]|_1 \|\sigma\| \qquad (10)\\ &= I(A:B|C)\|\sigma\|. \qquad (11) \end{aligned}$$

It is tempting to think that the same proof holds whenever all the reduced density matrices commute with each other. The statement itself is correct, but the preceding proof cannot be used. The reason why the proof worked for a classical systems was because there is a simultaneous local product basis for all the density matrices. It is possible for the density operators to commute with each other, yet do not allow a local product basis. For instance, one can easily prove that the reduced density matrix of 2D toric code commute with each other. However, the simultaneous eigenstates of these operators have entanglement in general between two different subsystems. Proof that includes such cases is presented in Appendix B. In fact, these are the very models we are interested in. We denote these models as *strongly commuting* models.

Definition 3. *H is strongly commuting if*

$$[\rho_A, \rho_B] = 0 \tag{12}$$

for all A, B. $\rho = \frac{e^{-\beta H}}{Z}$.

Applying the inequality from Appendix B, one can derive the following.

Lemma 2. *Suppose ρ is a Gibbs state of strongly commuting model. If nontrivial support of σ is contained in AC or BC,*

$$|Re(\mathcal{C}(\hat{H}_{A:B|C}, \sigma))| \leq I(A:B|C)\|\sigma\|. \tag{13}$$

III. MAIN RESULT

Consider a perturbation $V = \sum_i v_i$ and a strongly commuting hamiltonian $H = \sum_i h_i$ with $\|h_i\|, \|v_i\| \leq J$. We assume v_is and h_is are sufficiently local. One way to state this is to use interaction radius. We say interaction radius of v_i to be r if its nontrivial support is contained a ball with radius r centered at position i. We shall assume a finite radius of interaction r_H and r_V for the original hamiltonian and the perturbation. Main result can be written as the following.

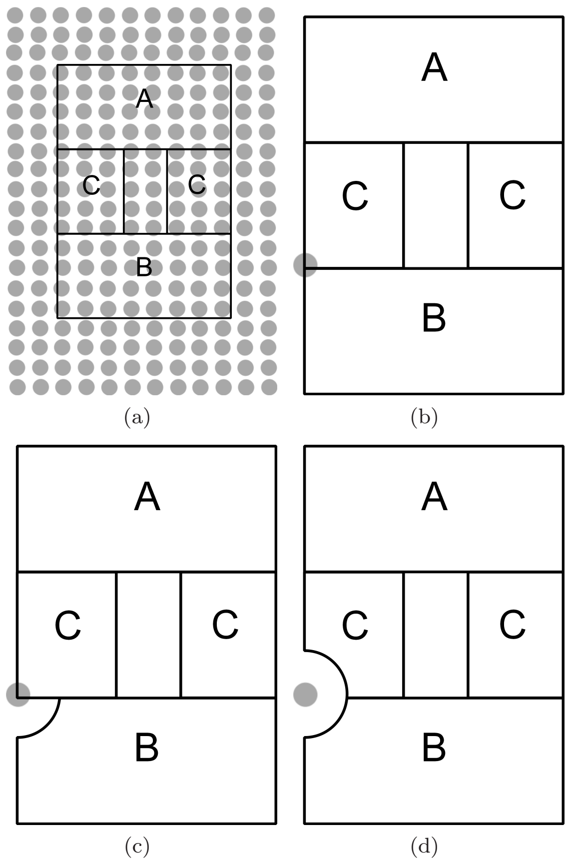


FIG. 8: Effect of the perturbation can be expressed as a sum of connected correlation function between local terms and $\hat{H}_{A:B|C}$. It is always possible to deform the order parameter away from any local operator.

Theorem 1. *If $\mathcal{S}(\{A_i\})$ is (l_1, l_2)-deformable for a strongly commuting hamiltonian,*

$$|\partial_H^V \mathcal{S}(\{A_i\})| \leq O(\max(f(l_1 - r_0), f(\frac{l_2}{4} - r_0))(\beta J Vol)^2), \tag{14}$$

where $r_0 = r_H + r_V$.

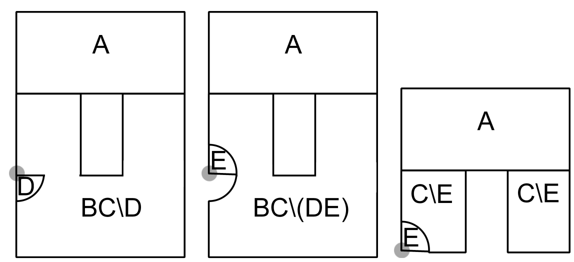
Proof. One may start with the following formula.

$$\partial_\rho^\sigma S_A = tr[-\sigma \ln \rho_A], \tag{15}$$

for $tr[\sigma] = 0$. Simple way to see this is to use Baker-Campbell-Hausdorff formula. Using the cyclic property of trace, one can show that the term in Eq.15 is the only surviving one. Using Eq.6, directional derivative can be expressed as a connected correlation function between entanglement spectrum of various subsystems and local perturbation.

For each of these local terms, we employ the following strategy. If the local term is sufficiently far away from the subsystems involved in computing the entropic order parameter, connected correlation function can be bounded by $\mathcal{C}(O_A, O_B) \leq \sqrt{2I(A:B)}\|O_A\|\|O_B\|$, where $\|\cdots\|$ denotes an operator norm. The operator norm of the linear combination of entanglement entropy can be bounded by $O(\beta J Vol)$. Operator norm of the local operator under transform $O_{ij} \rightarrow O_{ij} \frac{\tanh x_{ij}}{x_{ij}}$ is nonincreasing. Both statements about operator norm are nontrivial. See Appendix A for detailed explanation. $I(A:B)$ can be bounded by a function which depends on the distance between two subsystems.

If the local term is not sufficiently far away, one can always deform the linear combination of entanglement spectrum to be at least distance l_1 away from the local term. Connected correlation function between the deformed order parameter and local term can be bounded similarly. An example of such deformation is depicted in FIG.8.



(a)$\hat{H}_{A:D|BC\setminus D}$ (b)$\hat{H}_{A:E|BC\setminus(DE)}$ (c)$\hat{H}_{A:E|C\setminus E}$

FIG. 9: These three terms appear in a difference between the original and deformed order parameter.

The difference between $I(A:B|C)$ to $I(A:B'|C')$, is small in a thermodynamic limit. However, we still need to check if the correlation between the local observable and $\hat{H}_{A:B|C} - \hat{H}_{A:B'C'}$ is small. Terms arising from this difference is depicted in FIG.9. Depending on the position of the local term, there are two possibilities to consider. First, the support of the local observable is naturally included in the support of ABC, where the subsystems are used for computing $I(A:B|C)$. Since the distance between A and B is assumed to be at least l_2, for large enough l_2, the support of the local term either lies on AC or BC. Hence the correlation can be bounded using Lemma 2. It is also possible for the local term to have support outside of ABC. This is what happens in FIG.9. Without loss of generality, assume the local operator is closer to A than B. Enlarge A to AD such that A and D are connected. Using the chain rule $\hat{H}_{A:B|C} = \hat{H}_{AD:B|C} - \hat{H}_{D:C|AB}$, one can apply Lemma 2. Such correlation function can be bounded

Extra Assumption	Improvement
Superpolynomially decaying f	$Vol^2 \to Vol$
No thermal phase transition	$Vol^2 \to Vol \times Vol(S)$
Two assumptions combined	$Vol^2 \to Vol(S)$

TABLE I: A list of possible improvements on the bound depending on extra assumptions. The second and third means a combined effect of the superpolynomial decay of f and absence of thermal phase transition.

by $O(\beta^2 J^2 f(\frac{l_2}{4} - r_0) Vol)$. There are $O(Vol)$ such local terms. Combining all of these together, we arrive at the desired bound. □

As stated in the proof, our result applies to strongly commuting models. Interesting nontrivial example in this category is so called stabilizer models. For the proof of strong commuting condition, see Appendix C.

There are two length scales here. First length scale is the size of the system, which is encoded in the volume term of the bound. Second length scale is the size of the subsystems which are relevant to the computation of entropic order parameter. Hence one must scale the size of the subsystems appropriately to ensure the convergence of the bound. If f does not decay fast enough, such convergence may not be achieved. However, we are considering systems with finite correlation length. Therefore we expect f to decay exponentially fast, or at least superpolynomially.

Under such assumption, it is actually possible to reduce Vol^2 term to Vol. Note that the effect of local perturbations which are far away from the support of the order parameter was simply bounded by a product of the number of local terms and the maximal correlation. Instead of this naïve bound, one can imagine taking an integral over this space, so that the terms that are farther away can be bounded by a small number. Additional factor of Vol appears from the bound $\|\hat{H}_A\| \leq 2\beta\|H\|$, which is derived in Appendix A. If one assume an asymptotic conditional independence for all finite temperature, this term can be improved to $O(Vol(S))$, where S is a union of the subsystems used for computing the entropic order parameter.These potential improvements are summarized in Table I

IV. DISCUSSION

One can easily see that the proof of the main result is based on Lemma 1 and Lemma 2. This naturally leads us to the question of generalizing the analogous results when dropping the strong commuting condition. It is obvious Lemma 1 will not hold for general local hamiltonian models, but one can still ask if there exists a Lieb-Robinson type locality bound for such time evolution. Hastings showed such result for 1D system, but a similar statement can be established for any hamiltonian which allows nontrivial Lieb-Robinson bound.

Lemma 2 is really a statement about the spectrum of an operator $tr_A[\rho \hat{H}_{A:B|C} + \hat{H}_{A:B|C}\rho]$ in disguise. Numerical simulation shows that this operator may have negative eigenvalues, and existence of these negative eigenvalues forbids Lemma 2 to be generalized to arbitrary density operator. If one can bound the sum of these negative eigenvalues, it might be possible to establish an analogous result. We do not expect Lemma 2 to hold without the strong commuting condition, but a much looser bound is still applicable to the stability proof. For instance, $poly(\ln d)$ factor, where d is the dimension of the system, does not ruin the proof as long as the function f decays sufficiently fast.

For models which satisfy the strong commuting condition, our result resolves some of the confusing aspects of finite temperature topological entanglement entropy. At zero temperature, there are two well known ways of extracting topological entanglement entropy in 2D. Surprisingly at finite temperature, they show different behavior. Levin-Wen configuration produces a trivial topological entanglement entropy for a sufficiently large system size.[20] On the other hand, Kitaev-Preskill configuration produces a nonzero topological entanglement entropy that changes in temperature.[21] This seems puzzling; two different order parameters with same prediction at zero temperature give rise to different answers at finite temperature.

In the context of deformable entropic order parameter, this can be understood in a following way. First, Levin-Wen construction give rise to a (l_1, l_2)-deformable entropic order parameter for sufficiently large l_1 and l_2. The same statement does not hold for Kitaev-Preskill configuration. The main reason is the existence of triple point where subsystems A, B, and C meet. To see this, let us briefly review the stability proof. Finite temperature topological entanglement entropy of Kitaev-Preskill configuration depends on the temperature, so let us consider a temperature perturbation $\beta \to \beta + \delta\beta$. This corresponds to simply perturbing the coupling constant of the entire hamiltonian.

For the local perturbation terms that are sufficiently far away from the triple point, it is possible to deform the subsystems sufficiently far away while the deformation can be bounded by a small number. Hence the topological entanglement entropy will be stable against local terms far away from the triple point. However, one cannot find such deformation for the terms near the triple point. Assuming finite correlation length, there might be a $O(1)$ term that may not be bounded unless there is an extra insight to get rid of it. We suspect this is the origin of $O(1)$ term for 2D toric code at finite temperature.[21] The same logic can be applied generally.

This is an evidence that the configurations that can extract topological entanglement entropy at zero temperature may not be a suitable choice at finite temperature. Assuming such cancelation occurs for a set of subsystems $\{A_i\}$, these phenomena occur whenever there is a

triple point that at least three of these subsystems meet together. For this reason, we expect stacking Kitaev-Preskill type construction will not be a meaningful order parameter that can detect topological phase. Higher dimensional generalization of Levin-Wen construction seems to avoid this problem, as evidenced by the examples presented in this paper.

V. CONCLUSION

We presented a general formalism to prove a stability of finite temperature topological entanglement entropy under hamiltonian perturbation in a general context. When applied to most of the exactly solvable models exhibiting topological order at finite temperature, it shows that the topological entanglement entropy is invariant under a sum of geometrically local bounded-norm terms in a thermodynamic limit. We also presented a general reason to believe why certain topological entanglement entropy at zero temperature may not be stable against hamiltonian perturbation at finite temperature.

The technical foundation of this work is based on two statements. First statement concerns a certain variant of Lieb-Robinson type bound in quantum many body systems. Second statement concerns the spectrum of some operator whose trace is conditional mutual information. The first statement in fact holds generally for any quantum many-body system which satisfies Lieb-Robinson bound. Generalization of second statement seems elusive at this point. This is the only missing link to proving the stability of topologically invariant order parameters at finite temperature. Whether such statement is true or not remains as an open problem.

Another interesting question concerns the condition we imposed on the state, which is the asymptotic conditional independence. Circuit definition of topological order in rough terms, assumes a local indistinguishability of topologically different sectors. It is clear that the asymptotic conditional independence does not imply the circuit definition of topological order, since the 2D finite temperature examples presented in this paper are not topologically ordered. It would be interesting to see if the circuit definition of topological order implies asymptotic conditional independence.

Acknowledgments

This research was supported in part by NSF under Grant No. PHY-0803371, by ARO Grant No. W911NF-09-1-0442, and DOE Grant No. DE-FG03-92-ER40701. I thank Steve Flammia, Jeongwan Haah, and Spyridon Michalakis for helpful discussions.

Appendix A: Useful facts from Matrix Algebra

Let $\mathcal{H}_n$ be a n-dimensional Hilbert space. A set of operators acting on this Hilbert space is denoted as $\mathcal{B}(\mathcal{H}_n) : \mathcal{H}_n \to \mathcal{H}_n$. We are interested in a specific set of superoperators $\mathcal{S}(\mathcal{H}_n) : \mathcal{B}(\mathcal{H}_n) \to \mathcal{B}(\mathcal{H}_n)$ that share a common property.

A family of superoperators we study are the ones characterized by a function $f : \mathbb{R}^d \to \mathbb{C}, f(0) = 1$ that maps an operator O in a following way.

$$O_{ij} \to O_{ij} f(x_i - x_j) \tag{A1}$$

where i, j corresponds to the row and column of the matrix and $E_i \in (-\infty, \infty)$. The main question is to find the necessary and sufficient condition for such map to be a quantum operation. One can easily see that the trace is preserved by the the property $f(0) = 1$. It is tempting to think that such maps are quantum operation, since the entries of the matrices contract under the map. However, this is not necessarily true. For instance, consider a map with $f(x_i - x_j) = f(x_j - x_i) = 0$ and $f(x) = 1$ everywhere else. One can easily check that it maps a matrix $O_{ij} = 1 \quad \forall i, j = 1, 2, 3$, which has eigenvalue $3, 0, 0$ to another matrix that has negative eigenvalue.

A systematic way of deciding if the map is a quantum operation or not, is to use Choi's theorem on completely positive maps. Applied to these superoperators, Choi's theorem tells us that our map is completely positive if and only if the following matrix is positive semi-definite.

$$F_{ij} = f(x_i - x_j) \tag{A2}$$

It was Bochner who discovered a necessary and and sufficient condition for the positivity of such matrices. In our setting, Bochner's theorem implies the following statement.[22]

Theorem 2. *(Bochner)Let $f : \mathbb{R}^d \to \mathbb{C}$ be a continuous function. F_{ij} is positive semi-definite if and only if there exists a positive function g such that*

$$f(x) = \int e^{2\pi i p \cdot x} g(p) d^d p. \tag{A3}$$

This is the standard formulation of Bochner's theorem, but the same line of standard proof holds even when $g(p)$ is a distribution as long as it is positive.[22] Easiest example to see this is the unitary operation. The corresponding function is $f(x) = e^{ix}$. The inverse Fourier transform of this function is a delta function.

Let us carry on with another example. Consider $f(x) = \frac{x}{\sinh(x)}$. The inverse fourier transform is $g(p) = \frac{\pi^{3/2} \operatorname{sech}^2(\frac{\pi p}{2})}{2\sqrt{2}}$. This may seem like a contrived example, but a quantum operation associated with this function naturally arises when one studies the entanglement spectrum of a finite temperature equilibrium state(Gibbs state). Suppose we have a hamiltonian H. Gibbs state at

finite temperature is $\rho = \frac{e^{-H}}{Z}$, where $Z = tr(e^{-H})$. Typically one writes βH in the place of H, where β being the inverse temperature. However, we shall just absorb it into a definition of H for notational convenience. We would like to study the effect of perturbation $H \to H+\epsilon V$ on the entanglement spectrum. One can easily check directional derivative of ρ_A can be computed in a following way.

$$-\partial_H^V tr_{A^c}[e^{-H}] = tr_{A^c}[\int_0^1 e^{-sH} V e^{-(1-s)H}] \quad (A4)$$
$$= tr_{A^c}[e^{-\frac{1}{2}H} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-sH} V e^{sH} e^{-\frac{1}{2}H} ds] \quad (A5)$$
$$= tr_{A^c}[e^{-\frac{1}{2}H} \tilde{\Phi}(V) e^{-\frac{1}{2}H}], \quad (A6)$$

where we used Duhamel's formula on the first line. From the first line to the second line we simply rearranged the terms. On the third line, $\tilde{\Phi}(V) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-sH} V e^{sH} ds$. If we expand V in terms of the eigestates of H, $\tilde{\Phi}$ maps $V_{ij} \to V_{ij}\frac{\sinh(x)}{x}$, where $x = \frac{E_i - E_j}{2}$ and E_i is the ith energy eigenstate. In other words, $\tilde{\Phi}$ is an inverse of a quantum operation $\Phi(V)_{ij} = V_{ij}\frac{x}{\sinh(x)}$. As one can see from this example, one can define a set of superoperators defined by a hamiltonian H and a function $f(x)$. We shall denote such operation and its inverse as following.

Definition 4.

$$\Phi^H_{f(x)} : O_{ij} \to O_{ij} f(E_i - E_j) \quad (A7)$$
$$\tilde{\Phi}^H_{f(x)} : O_{ij} \to O_{ij} \frac{1}{f(E_i - E_j)} \quad (A8)$$

$O_{ij} = \langle i| O |j\rangle$, where $|i\rangle$ is the ith eigenstate of H and E_i is the corresponding eigenvalue.

From these results, we can show the following lemma.

Lemma 3. *$\Phi^H_{f(x)}$ is a quantum operation if and only if $f(x)$ has a positive inverse Fourier transform.*

Corollary 1.

$$\|\Phi^H_{\tanh x/x}(V)\| \leq \|V\|, \quad (A9)$$

Proof. It follows from following two identities.

$$\frac{\tanh x}{x} = \sum_{k=0}^{\infty} \frac{2}{x^2 + (k+\frac{1}{2})^2\pi^2}. \quad (A10)$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} e^{-ipx} dx = \sqrt{\frac{\pi}{2}} e^{-|p|}. \quad (A11)$$

□

Lemma 4. *For $\|H\|, \|V\| < \infty$,*

$$\partial_\beta \hat{H}_A = \langle H\rangle - \Phi_f^{\hat{H}^A} \langle H\rangle_{A^c|A}, \quad (A12)$$

where $f(x) = \frac{x/2}{\sinh(x/2)}$. $\langle O\rangle_{A^c|A} = \rho_A^{-\frac{1}{2}} tr_{A^c}[\rho^{\frac{1}{2}} O \rho^{\frac{1}{2}}]\rho_A^{-\frac{1}{2}}$ is a quantum conditional expectation.

Corollary 2.

$$\|\hat{H}_A\| \leq 2\beta\|H\| \quad (A13)$$

Proof. $\Phi_f^{\hat{H}_A}$ is a quantum operation. Conditional expectation is also a quantum operation.[23] Using the norm contractivity of quantum operation, one can easily obtain $\|\hat{H}_A\| \leq 2\beta\|H\|$. □

In the absence of finite temperature phase transition, one can expect an existence of finite length scale at all finite temperature. In this case, it is possible to cancel out most of the correlations residing far away from A. We can get a better bound this way, but for the purpose of the paper this is not so crucial.

Appendix B: Modification of Klein's inequality

Klein's inequality states the following statement.

Theorem 3. *For $A, B > 0$,*

$$tr[A(\ln A - \ln B)] \geq tr[A - B]. \quad (B1)$$

What we would like to prove is an analogous statement for partial trace.

Lemma 5. *For $A, B > 0$, $[A, B] = 0$,*

$$tr_C[A(\ln A - \ln B)] \geq tr_C[A - B] \quad (B2)$$

Proof. Let $|i\rangle$ denote a common eigenstate of A and B. a_i and b_i denote the corresponding eigenstates. $\rho_i = tr_C[|i\rangle\langle i|]$.

$$tr_C[A(\ln A - \ln B)] = \sum_i a_i(\ln\frac{a_i}{b_i})\rho_i \quad (B3)$$
$$\geq \sum_i a_i(1 - \frac{b_i}{a_i})\rho_i \quad (B4)$$
$$= tr_C[A - B]. \quad (B5)$$

From the first two the second line, we used $\ln\frac{1}{x} \geq 1 - x$. □

Corollary 3. *For the Gibbs state of strongly commuting hamiltonian,*

$$tr_A[\rho_{ABC}(\hat{H}_{ABC} + \hat{H}_B - \hat{H}_{BC} - \hat{H}_{AB})] \geq 0. \quad (B6)$$

Appendix C: Strongly Commuting Models

Strongly commuting condition seems to be a very strong constraint, but fortunately there are number of interesting models that satisfy this condition. First class of models we would like to consider are models based on quantum error correcting code. Stabilizer code S is an abelian subgroup of a generalized pauli group $\mathcal{P}^n$ that does not contain $-I$ as a group element. If there exist a spatially local group generators for S, one can associate it to a quantum many-body system with hamiltonian corresponding to those generators. Since the group is abelian, the local generators commute with each other. Furthermore, all the elements of the group is hermitian, so the corresponding hamiltonian is simply a sum of the generators. Furthermore, since the stabilizer group elements are vanishes under any partial trace over its nontrivial support. Using this fact, one can prove the following.

Lemma 6. *$\rho_A = \sum_{S_i \in S(A)} c_i S_i$ for some coefficients $\{c_i\}$.*

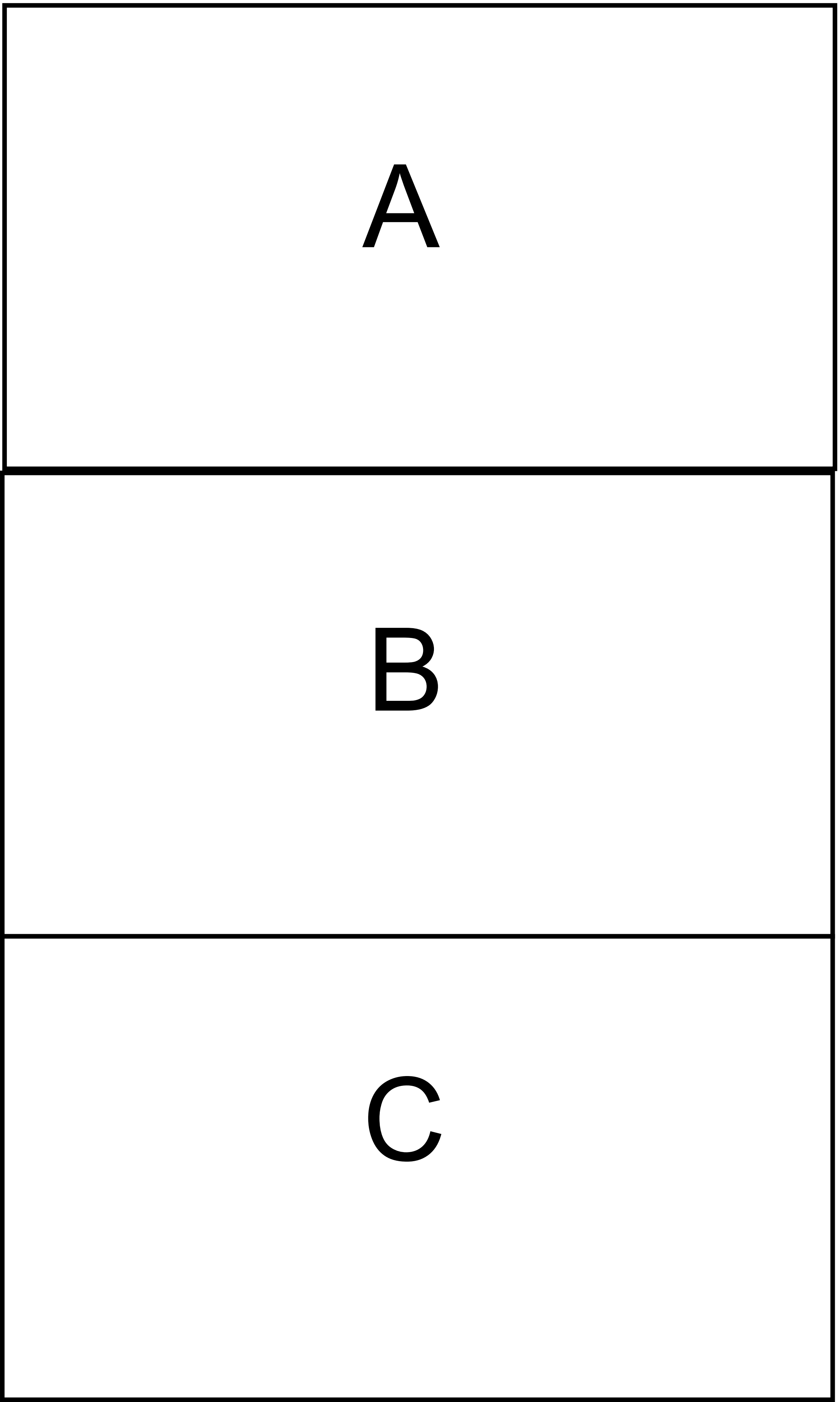
Proof. ρ can be expanded as a sum of stabilizer group elements. When taking the partial trace, any operator that has nontrivial support on A^c vanishes. The ones that have nontrivial support only on A survives, and they are again a stabilizer group element. Hence ρ_A can be written as a sum of elements in $S(A)$. □

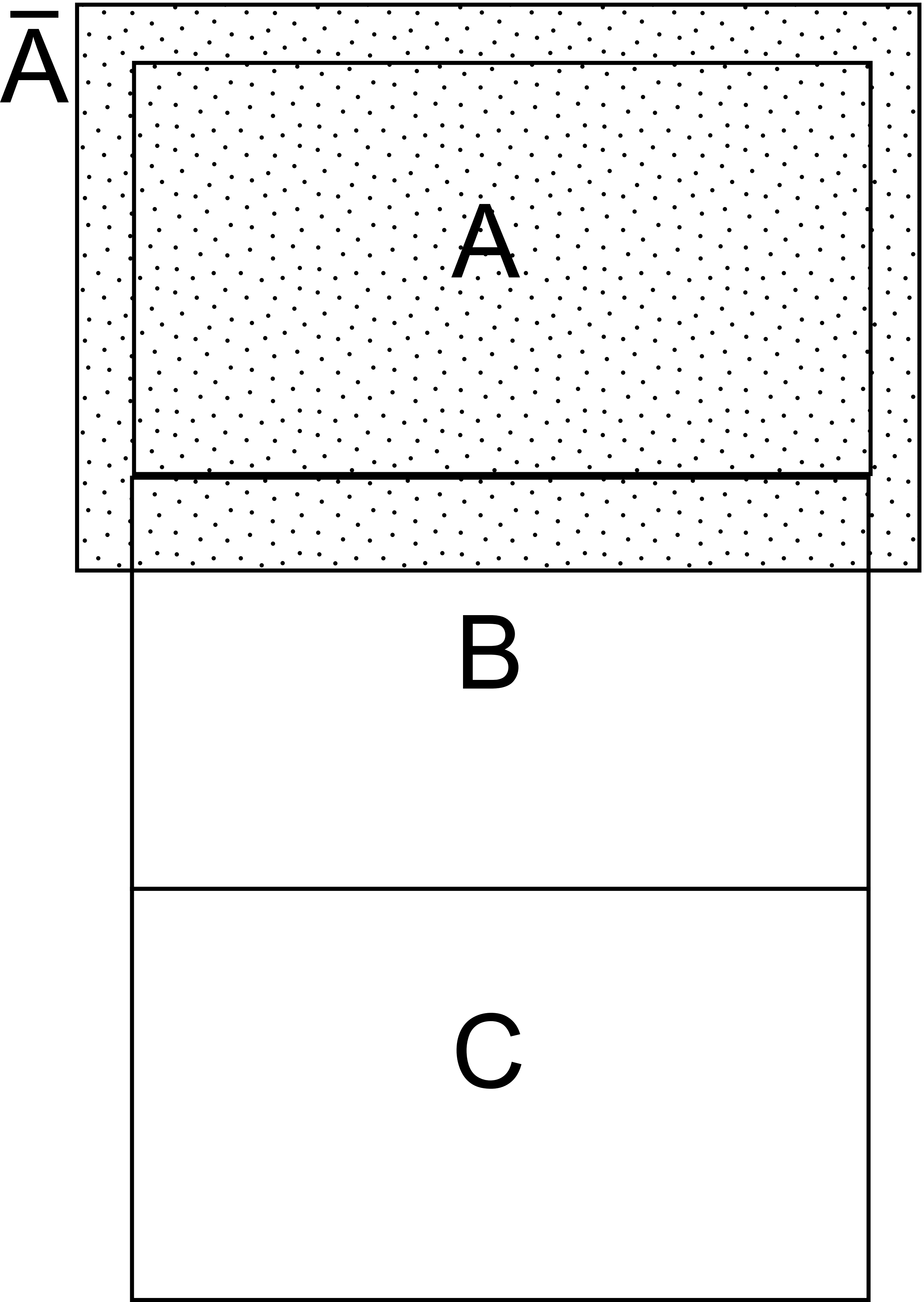
Corollary 4. *For Gibbs state of stabilizer hamiltonian,*

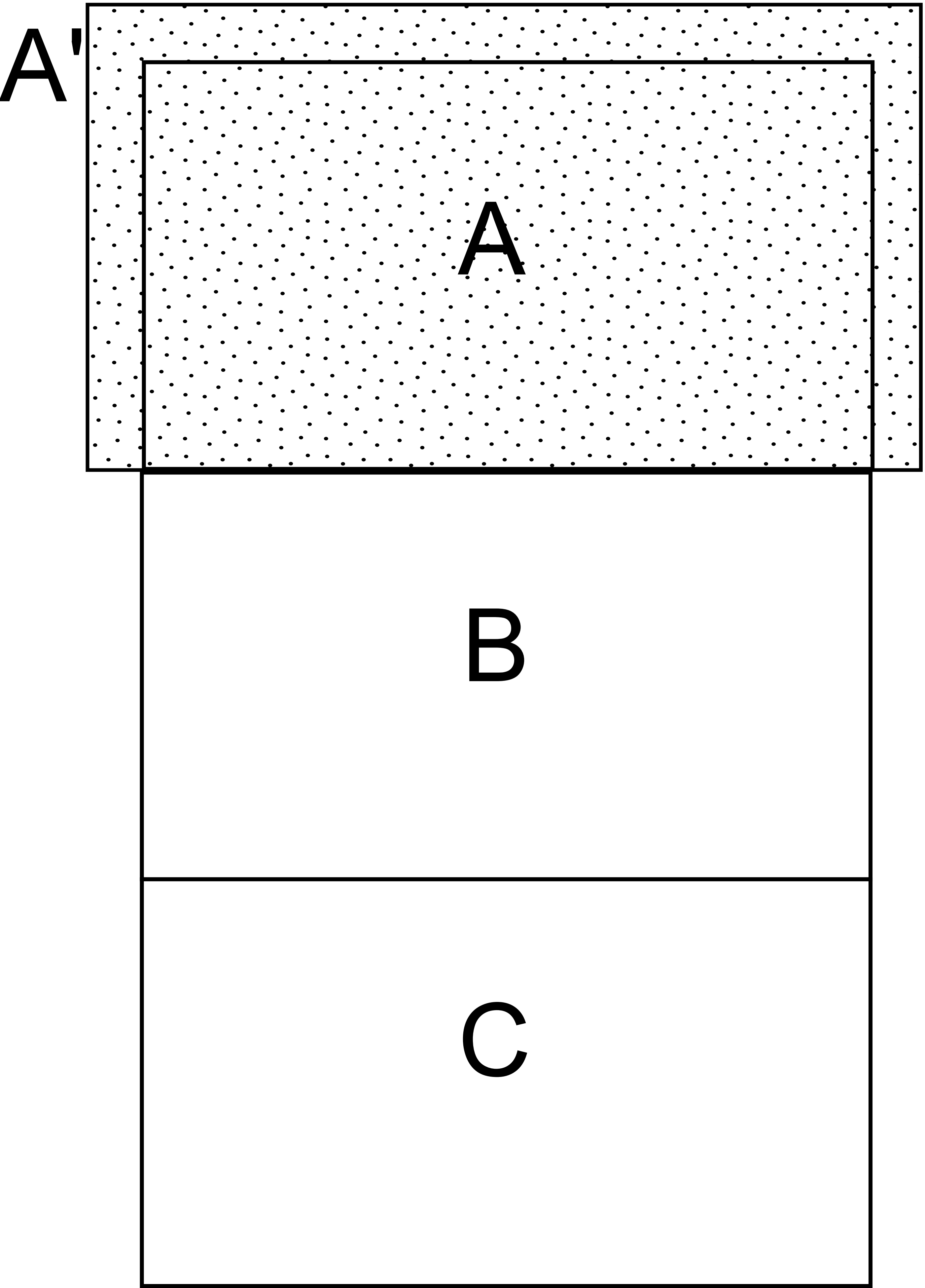
$$[\rho_A, \rho_B] = 0. \tag{C1}$$

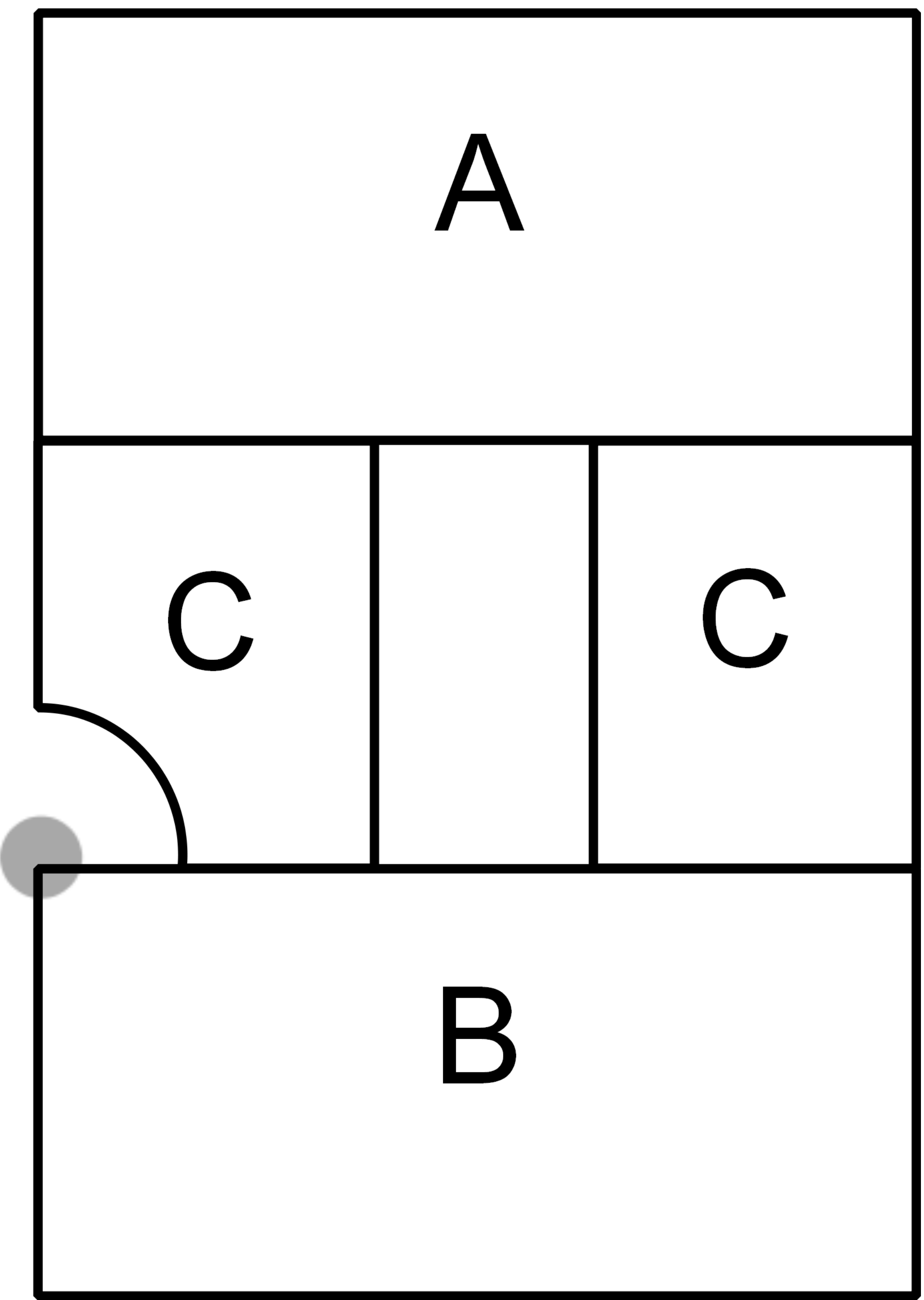
Stabilizer models cover interesting models in many different dimensions, including Toric code in 2,3,4 spatial dimension, topological color code, quantum glass code, and other variants of 3D models.

[1] A. Y. Kitaev, Annals Phys. **303**, 2 (2003), quant-ph/9707021.
[2] M. A. Levin and X.-G. Wen, Phys.Rev. B **71**, 045110 (2005), cond-mat/0404617.
[3] S. Bravyi, M. Hastings, and S. Michalakis, J. Math. Phys. **51**, 093512 (2010), 1001.0344.
[4] S. Bravyi and M. B. Hastings (2010), 1001.4363.
[5] S. Michalakis and J. Pytel (2011), 1109.1588.
[6] S. Bravyi and B. Terhal, New J. Phys. **11**, 043029 (2008), 0810.1983.
[7] S. Bravyi, D. Poulin, and B. Terhal, Phys. Rev. Lett. **104**, 050503 (2009), 0909.5200.
[8] J. Haah and J. Preskill (2010), 1011.3529.
[9] C. Castelnovo and C. Chamon, Phys. Rev. B **78,**, 155120 (2008), 0804.3591.
[10] H. Bombin and M. A. Martin-Delgado, Phys.Rev.Lett. **97**, 180501 (2006).
[11] H. Bombin and M. A. Martin-Delgado, Phys.Rev.B **75**, 075103 (2007).
[12] B. Yoshida (2011), 1103.1885.
[13] Z. Nussinov and G. Ortiz, Phys. Rev. B **77**, 064302 (2008).
[14] Z. Nussinov and G. Ortiz, Annals of Physics **324**, 977 (2009).
[15] I. H. Kim, Phys. Rev. A **83**, 052308 (2011).
[16] R. Alicki, M. Horodecki, P. Horodecki, and R. Horodecki, Open Syst. Inf. Dyn. **17**, 1 (2008), 0811.0033.
[17] M. B. Hastings, Phys. Rev. Lett. **93**, 126402 (2004).
[18] M. B. Hastings, Phys. Rev. B **76**, 201102 (2007).
[19] T. Grover, A. M. Turner, and A. Vishwanath (2011), 1108.4038.
[20] C. Castelnovo and C. Chamon, Phys. Rev. B **76,**, 184442 (2007), 0704.3616.
[21] S. Iblisdir, D. Perez-Garcia, M. Aguado, and J. Pachos, Nuclear Physics B **829**, 401 (2010).
[22] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vol. II* (Academic Press, 1975).
[23] M. S. Leifer and R. W. Spekkens (2011), 1107.5849.


A
B
C


Ā
A
B
C


A'
A
B
C


A
C
C
B